Problem 1

1. For $n \geq 1 \in \mathbb{N}$, consider the function

$$f_n(x) = \underbrace{\sqrt{x + \sqrt{x + \sqrt{\dots}}}}_{n \text{ times}}.$$

For example, $f_1(x) = \sqrt{x}$ and $f_2(x) = \sqrt{x + \sqrt{x}}$. For $n \geq 1$, find a formula for $f_{n+1}(x)$ in terms of $f_n(x)$.

2. Use the chain rule to find a formula for $f'_{n+1}(x)$ in terms of $f'_n(x)$.

3. Using your previous formula, write the explicit formulas for $f'_2(x)$ and $f'_3(x)$.

The **Mean Value Theorem** states that if a function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in [a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Problem 2

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere.

1. If $f'(x) = 0$ for all $x \in \mathbb{R}$, show that f is constant.

2. If $f'(x) > 0$ for all $x \in \mathbb{R}$, show that f is strictly increasing.

Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere and suppose f' is bounded. That is, there is $M \in \mathbb{R}$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous.[a]

Hint: show that if $x < y$, then $|f(x) - f(y)| \leq M|x - y|$.

[a] Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous when $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in \mathbb{R})[|x - y| < \delta \to |f(x) - f(y)| < \epsilon]$.

The **Inverse Function Theorem** states that if a function $f : \mathbb{R} \to \mathbb{R}$ is C^1 (differentiable with continuous derivative), and $f'(p) \neq 0$, then there exists open intervals U containing p and V containing $f(p)$ such that

$$\hat{f} : U \to V, \hat{f}(x) = f(x)$$

is bijective, and $\hat{f}^{-1}$ is C^1 with

$$(\hat{f}^{-1})'(y) = \frac{1}{\hat{f}'(\hat{f}^{-1}(y))}.$$

In other words, f is *locally* invertible, and its local inverse's derivative can be computed using the above formula.

Problem 4

Give an example of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(p) = 0$ yet f is still locally invertible. What is its derivative?

Problem 5

We define $\arctan : \mathbb{R} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ as the inverse of tan with its domain restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to ensure injectivity. Using the Inverse Function Theorem, find the derivative of arctan.

Problem 6
Consider the function

$$f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} x + x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

1. Find the derivative of f at 0.

2. Find the derivative of f elsewhere. This, combined with 1, shows that f is differentiable.

3. Show that

$$\lim_{x \to 0} f'(x) \neq f'(0).$$

   Conclude that f is not C^1.

It turns out that this function is not locally invertible: given any $\rho > 0$, the restriction of f to $(-\rho, \rho)$ is not injective. This is why it is necessary that we assume f is C^1 in the Inverse Function Theorem.